SOLUTION OF AN INVERSE PROBLEM OF HEAT CONDUCTION BY ITERATION METHODS

O. M. Alifanov

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An iteration scheme for determining nonstationary heat flux is constructed for the linear case. The scheme is based on the two gradient slope methods.

We consider the following inverse problem for the homogeneous linear equation of heat conduction in the region $\{0 < x < b,\ \tau > 0\}$. The heat flux function $q(\tau)$ at the left boundary of the region is to be determined from the known temperature $f(\tau)$ and the heat flux $Q(\tau)$ at the right boundary. Any problem of determination of a boundary function can be reduced to this form when $f(\tau)$ is known at an inner point $0 < x_1 < b$ and the boundary condition is specified at the point $x = b$. For this purpose it is necessary to solve the corresponding boundary value problem in the region $x_1 < x < b$.

Thus we have (it is assumed that the reduction to the initial condition has been made)

$$\frac{\partial T(\xi,\ \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{\partial^2 T(\xi,\ \mathrm{Fo})}{\partial \xi^2}, \quad 0<\xi<1, \quad \mathrm{Fo}>0, \tag{1}$$

$$T(\xi,\ 0) = 0, \tag{2}$$

$$-\frac{\lambda}{b}\,\frac{\partial T(1,\ \mathrm{Fo})}{\partial \xi} = Q(\mathrm{Fo}), \tag{3}$$

$$T(1,\ \mathrm{Fo}) = f(\mathrm{Fo}), \tag{4}$$

$$-\frac{\lambda}{b}\,\frac{\partial T(0,\ \mathrm{Fo})}{\partial \xi} = q(\mathrm{Fo}) - ?, \tag{5}$$

where $\mathrm{Fo} = a\tau/b^2$, $\xi = x/b$.

We interpret the incorrectly formulated problem (1)-(5) as a problem of optimum control, i.e., to choose the control q(Fo) from the condition of minimum deviation of $T(q(\mathrm{Fo}),\ 1,\ \mathrm{Fo}) = T^*(\mathrm{Fo})$ from the given function f(Fo) in the metric of space L_2 (square integrable functions):

$$J(q) = \int_0^{\mathrm{Fo}_m} [T^*(\mathrm{Fo}) - f(\mathrm{Fo})]^2\, d\mathrm{Fo} - \min_q. \tag{6}$$

We shall solve the formulated extremum problem by the gradient methods. For this purpose we write out the formula for the gradient of functional (6).

Let us assume that q(Fo) undergoes an increment $\Delta q(\mathrm{Fo})$. Then the temperature $T(\xi,\ \mathrm{Fo})$ changes by an amount $\Delta T(\xi,\ \mathrm{Fo})$ which satisfies the conditions

$$\frac{\partial \Delta T(\xi,\ \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{\partial^2 \Delta T(\xi,\ \mathrm{Fo})}{\partial \xi^2}, \quad 0<\xi<1, \quad \mathrm{Fo}>0, \tag{7}$$

$$\Delta T(\xi,\ 0) = 0, \tag{8}$$

$$\frac{\partial \Delta T(1,\ \mathrm{Fo})}{\partial \xi} = 0, \tag{9}$$

$$-\frac{\lambda}{b}\,\frac{\partial \Delta T(0,\ \mathrm{Fo})}{\partial \xi} = \Delta q(\mathrm{Fo}). \tag{10}$$

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The functional J gets the corresponding increment

$$\Delta J \equiv J(q+\Delta q) - J(q) = J(T+\Delta T) - J(T).$$

Hence neglecting quantities of the second order of smallness we obtain

$$\Delta J = \int_0^{\mathrm{Fo}_m} 2[T^*(\mathrm{Fo}) - f(\mathrm{Fo})]\, \Delta T(1, \mathrm{Fo})\, d\,\mathrm{Fo}. \tag{11}$$

Following [1] we write the boundary value problem conjugate to (7)-(10):

$$\frac{\partial \psi(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}} = -\frac{\partial^2 \psi(\xi, \mathrm{Fo})}{\partial \xi^2}, \quad 0<\xi<1, \quad 0<\mathrm{Fo}<\mathrm{Fo}_m, \tag{12}$$

$$\psi(\xi, \mathrm{Fo}_m) = 0, \tag{13}$$

$$\frac{\partial \psi(1, \mathrm{Fo})}{\partial \xi} = 2[T^*(\mathrm{Fo}) - f(\mathrm{Fo})], \tag{14}$$

$$\frac{\partial \psi(0, \mathrm{Fo})}{\partial \xi} = 0. \tag{15}$$

Then for the increment of functional (11) we get

$$\Delta J = \int_0^{\mathrm{Fo}_m} \Delta T(\xi, \mathrm{Fo}) \frac{\partial \psi(\xi, \mathrm{Fo})}{\partial \xi}\, d\,\mathrm{Fo}\,\Bigg|_{\xi=0}^{\xi=1} = \int_0^1 d\xi \frac{d}{d\xi} \int_0^{\mathrm{Fo}_m} \Delta T(\xi, \mathrm{Fo}) \frac{\partial \psi(\xi, \mathrm{Fo})}{\partial \xi}\, d\,\mathrm{Fo}$$

from which we obtain

$$\Delta J = \int_0^1 I_1(\xi)\, d\xi + \int_0^1 I_2(\xi)\, d\xi, \tag{16}$$

where

$$I_1(\xi) = \int_0^{\mathrm{Fo}_m} \Delta T(\xi, \mathrm{Fo}) \frac{\partial^2 \psi(\xi, \mathrm{Fo})}{\partial \xi^2}\, d\,\mathrm{Fo},$$

$$I_2(\xi) = \int_0^{\mathrm{Fo}_m} \frac{\partial \Delta T(\xi, \mathrm{Fo})}{\partial \xi} \frac{\partial \psi(\xi, \mathrm{Fo})}{\partial \xi}\, d\,\mathrm{Fo}$$

Next making use of the conjugate equation (12) and integrating by parts with (8) and (13) taken into consideration, we have

$$I_1(\xi) = -\int_0^{\mathrm{Fo}_m} \Delta T(\xi, \mathrm{Fo}) \frac{\partial \psi(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}}\, d\,\mathrm{Fo} = \int_0^{\mathrm{Fo}_m} \psi(\xi, \mathrm{Fo}) \frac{\partial \Delta T(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}}\, d\,\mathrm{Fo}.$$

Changing the order of integration and passing on to the spatial derivative $\Delta T(\xi, \mathrm{Fo})$ from (7) we obtain

$$\int_0^1 I_1(\xi)\, d\xi = \int_0^{\mathrm{Fo}_m} d\,\mathrm{Fo} \int_0^1 \psi(\xi, \mathrm{Fo}) \frac{\partial^2 \Delta T(\xi, \mathrm{Fo})}{\partial \xi^2}\, d\xi.$$

Integrating by parts with the use of (9) and (10) we get

$$\int_0^1 I_1(\xi)\, d\xi = \frac{b}{\lambda} \int_0^{\mathrm{Fo}_m} \psi(0, \mathrm{Fo})\, \Delta q(\mathrm{Fo})\, d\,\mathrm{Fo} - \int_0^1 I_2(\xi)\, d\xi. \tag{17}$$

From (16) and (17) we get

$$\Delta J = \frac{b}{\lambda} \int_0^{\mathrm{Fo}_m} \psi(0, \mathrm{Fo})\, \Delta q(\mathrm{Fo})\, d\,\mathrm{Fo}.$$

In Hilbert space L_2 [1]

$$\Delta J = \int_0^{\mathrm{Fo}_m} J'(q)\,\Delta q(\mathrm{Fo})\,d\,\mathrm{Fo}.$$

Thus the expression for the gradient of functional (6) is

$$J'(q) = \frac{b}{\lambda}\,\psi(0,\ \mathrm{Fo}). \tag{18}$$

For constructing the iteration approximations to the desired heat flux function $q(\tau)$ we shall use the methods of steepest descent and conjugate gradients.

In the first case starting from a certain "test" solution $q^0(\mathrm{Fo})$ the subsequent approximations are found from the formula

$$q^{k+1} = q^k - \beta_k J'(q^k), \qquad k = 0,\ 1,\ \ldots \tag{19}$$

The coefficient β_k, which determines the step in going from q^k to q^{k+1}, is found from the condition of minimum of $J(q^{k+1})$:

$$\min_\beta \int_0^{\mathrm{Fo}_m} [T(q^k - \beta_k J'^k) - f]^2\,d\,\mathrm{Fo} = \min_\beta \int_0^{\mathrm{Fo}_m} [-\beta\Delta T(J'^k) + T(q^k) - f]^2\,d\,\mathrm{Fo}.$$

Hence we obtain

$$\beta_k = \frac{\int_0^{\mathrm{Fo}_m} \Delta T(J'^k)\,[T(q^k) - f]\,d\,\mathrm{Fo}}{\int_0^{\mathrm{Fo}_m} \Delta T^2(J'^k)\,d\,\mathrm{Fo}}.$$

In using the method of conjugate gradients [2] the sequence $\{q^k\}$ is sought in the following form:

$$q^{k+1} = q^k - \beta_k p^k, \qquad k = 0,\ 1,\ \ldots, \tag{20}$$

where

$$p^k = J'^k + \gamma_k p^{k-1},$$

$$\gamma_k = \frac{\int_0^{\mathrm{Fo}_m} [J'^k(\mathrm{Fo})]^2\,d\,\mathrm{Fo}}{\int_0^{\mathrm{Fo}_m} [J'^{k-1}(\mathrm{Fo})]^2\,d\,\mathrm{Fo}} \qquad (\gamma_0 = 0).$$

The coefficient β_k is found from the condition

$$\min_\beta J(q^k - \beta p^k).$$

According to the superposition principle the solution of the second boundary value problem for the linear homogeneous heat conduction equation with boundary functions $q_1(\mathrm{Fo})$ and $q_2(\mathrm{Fo})$ can be written in the form

$$\Theta(\xi,\ \mathrm{Fo}) = \int_0^{\mathrm{Fo}} q_2(\eta)\,\frac{\partial\vartheta(\xi,\ \mathrm{Fo} - \eta)}{\partial\,\mathrm{Fo}}\,d\eta + \int_0^{\mathrm{Fo}} q_1(\eta)\,\frac{\partial\vartheta(1 - \xi,\ \mathrm{Fo} - \eta)}{\partial\,\mathrm{Fo}}\,d\eta,$$

where $\vartheta(\mathrm{Fo})$ is the solution of problem (1), (2), (3), (5) with unit heat flux at one boundary of the plate and zero at the other.

Using a suitable approximation method [3] we obtain the following formula for computing $\Theta(\xi)$ at the n-th instant of time:

$$\Theta_n(\xi) = \sum_{i=1}^{n} \bar{q}_{2i}\vartheta^m_{1(m-n+i)}(\xi) + \sum_{i=1}^{n} \bar{q}_{1i}\vartheta^m_{2(m-n+i)}(\xi), \quad n = 1,\ 2,\ \ldots\ m, \tag{21}$$

where

$$\bar{q}_{1i} = \frac{b}{2\lambda}(q_{1i-1} + q_{1i}),\ \ \bar{q}_{2i} = \frac{b}{2\lambda}(q_{2i-1} + q_{2i}).$$

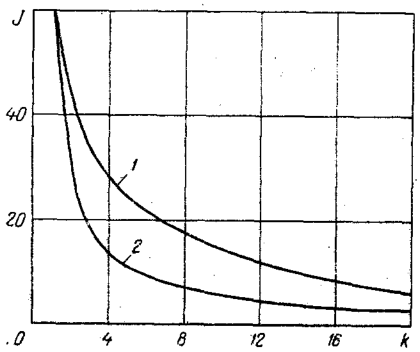

Fig. 1. Example of convergence of the iteration sequences (J, deg^2, ΔFo = 0.04, q^0(Fo) = 0, f_n, unperturbed values): 1) computation of approximations by formula (19); 2) by formula (20).

Below we shall need $\Theta_n(0)$ and $\Theta_n(1)$. In this case

$$\vartheta_{1i}^m(0) = -2\sqrt{\Delta \mathrm{Fo}(m-p)} \sum_{j=0}^{N_1} \left(i\Phi^*\left[\frac{j}{\sqrt{\Delta \mathrm{Fo}(m-p)}}\right] + i\Phi^*\left[\frac{j+1}{\sqrt{\Delta \mathrm{Fo}(m-p)}}\right]\right)\Bigg|_{p=i-1}^{p=i},$$

$$\vartheta_{2i}^m(0) = -4\sqrt{\Delta \mathrm{Fo}(m-p)} \sum_{j=0}^{N_2} i\Phi^*\left[\frac{2j+1}{2\sqrt{\Delta \mathrm{Fo}(m-p)}}\right]\Bigg|_{p=i-1}^{p=i},$$

$$\vartheta_{1i}^m(1) = \vartheta_{2i}^m(0),$$

$$\vartheta_{2i}^m(1) = \vartheta_{1i}^m(0),$$

where

$$\Delta \mathrm{Fo} = \frac{a\tau_m}{b^2 m},$$

$$i\Phi^*[u] = \frac{1}{\sqrt{\pi}}\exp[-u^2] - u(1-\Phi[u]) \quad (i\Phi^*[\infty] = 0),$$

and $\Phi[u] = 2/\sqrt{\pi}\int_0^u \exp[-\eta^2]d\eta$ is the error integral.

The numbers N_1 and N_2 are chosen from the conditions

$$i\Phi^*\left[\frac{N_1}{\sqrt{\Delta \mathrm{Fo}(m-p)}}\right] + i\Phi^*\left[\frac{N_1+1}{\sqrt{\Delta \mathrm{Fo}(m-p)}}\right] \leqslant \varepsilon,\ i\Phi^*\left[\frac{2N_2+1}{2\sqrt{\Delta \mathrm{Fo}(m-p)}}\right] \leqslant \varepsilon,$$

where $\varepsilon > 0$ is a prespecified small quantity.

In constructing the sequence of elements q^k according to (19) or (20) in each iteration it is necessary to solve three problems of determination of the temperature T*(Fo), the function ψ(0, Fo), and the increment $\Delta T(u^k)$ ($u = J'^k$ or p^k). All these quantities can be found from (21) with corresponding values of ξ, q_1, and q_2. For T_n^{*k} we have

$$\xi = 1,\quad \bar{q}_{1n} = \bar{q}_n^k\left(\bar{q}_n^0 = \frac{b}{2\lambda}(q_{n-1}^0 + q_n^0)\right),\quad \bar{q}_{2n} = \frac{b}{2\lambda}(Q_{n-1} + Q_n);$$

for $\psi_n^k(0)$

$$\xi = 0,\quad \bar{q}_{1n} = 0,\quad \bar{q}_{2n} = 2\bar{\varkappa}_n^k$$

$$\left(\bar{\varkappa}_n^k = \frac{1}{2}(\varkappa_{n-1}^k + \varkappa_n^k),\quad \varkappa_n^k = T_{m-n}^{*k} - f_{m-n}^k\right);$$

for $\Delta T_n^k(u)$

$$\xi = 1,\quad \bar{q}_{1n} = u_n^k,\quad \bar{q}_{2n} = 0\ (\Delta T_0^k = 0).$$

From these quantities we compute

$$J_n'^k(\bar{q}) = \psi_{m-n}^k(0),$$

$$\beta_k \simeq \frac{\sum_{n=1}^{m}(F_{n-1}^k + F_n^k)}{\sum_{n=1}^{m}[(\Delta T_{n-1}^k)^2 + (\Delta T_n^k)^2]},$$

$$\gamma_k \simeq \frac{\sum_{n=1}^{m}[(J_{n-1}'^k)^2 + (J_n'^k)^2]}{\sum_{n=1}^{m}[(J_{n-1}'^{k-1})^2 + (J_n'^{k-1})^2]} \quad (\gamma_0 = 0),$$

where $F_n^k = \Delta T_n^k \bar{\varkappa}_n^k$.

The gradient methods of solving the inverse problem of heat conduction (1)-(5) were verified taking a series of methodological examples and showed good results. As expected [2] the sequence (20) converges

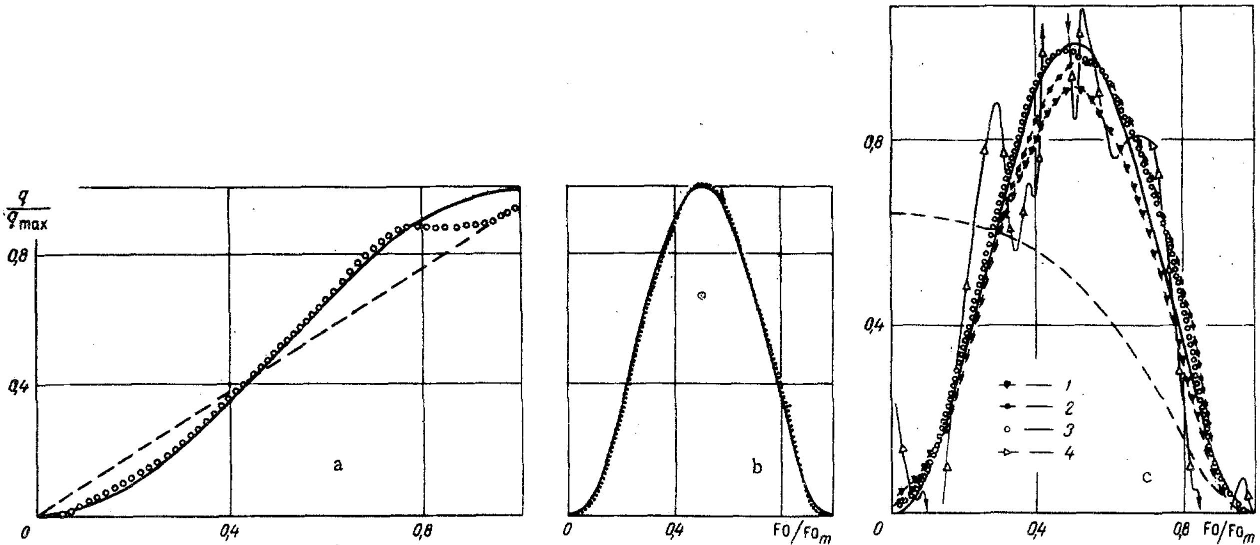

Fig. 2. Reconstruction of heat fluxes by the method of conjugate gradients (q_{max} is the largest value of the desired function q(Fo)): a) $\Delta F = 0.01$; f_n, unperturbed values; continuous curve) exact solution; dashes) initial approximation; points) 25th iteration; b) $\Delta Fo = 0.005$, $q^0(Fo) = 0$, fn, unperturbed values; continuous curve) exact solution; dashes) 11th iteration; c) $\Delta Fo = 0.03$, $q^0(Fo) = 0$, f_n, unperturbed values; continuous curve) exact solution; dashes) 1st iteration; 1) second iteration; 2) 3rd iteration; 3) approximation chosen according to discrepancy principle (4th iteration); 4) 50th iteration.

somewhat faster and permits a closer approach to the minimum of functional (6) compared to (19). This is illustrated in Fig. 1 for one of the examples.

We note that a disadvantage of the above algorithm of determining the heat flux is the absence of a uniform convergence of the approximations (19) and (20) (the convergence occurs only in L_2). Because of condition (13) the gradient of the functional J'(q) deviates from the exact value in some neighborhood of the end point $Fo = Fo_m$. This causes a distortion of the solution q(Fo) also in this neighborhood ($q(Fo_m) = q^0(Fo_m)$, $q'(Fo_m) = q^{0'}(Fo_m)$) (see Fig. 2a).

However, if the behavior of the function q(Fo) near the right boundary of the time interval is predicted beforehand, then the proposed algorithm is very efficient. For example, if it is known that the desired heat flux vanishes at $Fo = Fo_m$, then specifying the initial approximation $q^0(Fo)$ we obtain q(Fo) very close to the exact function (Fig. 2b).

The solution of the inverse problem is quite stable to perturbations of the input data both in the case of the method of steepest descent and the method of conjugate gradients. This is mainly accounted for by a significant decrease of the rate of convergence of these methods of gradient slope with the increase in the number of iterations which in a way damps the "oscillations" of the approximations $q^k(Fo)$. The results for the first three iterations obtained from formula (20) are shown in Fig. 2c for sufficiently large fluctuation perturbations of the input data ($f_{\delta n} = f_n + \delta \zeta_n$, where $\delta = 10\% f_{max}$; ζ_n is a random quantity uniformly distributed in the segment [—1, 1]. The last approximation is very close to the sought curve of q(Fo).

If the process of approximations is continued further, the results change very insignificantly in the next three iterations. A gradual oscillation of the solution is observed starting from 7-8-th iteration (50-th approximation for this example is shown in Fig. 2c).

Computations show that it is advisable to use the discrepancy principle [4, 5] for terminating the iteration process. Here the number of iterations plays the role of the regularization parameter. If necessary the exit to the given discrepancy level is obtained by a suitable choice of the parametric step β. If the input data f_n are sufficiently accurate, then for the above algorithm it is generally always possible to specify beforehand the number of iterations (10-15) which would correspond to a reasonable proximity of the last approximation to the actual heat flux function.

Apart from its high accuracy this algorithm is also economical in respect of computation time and compared to the algorithms of [6, 7] it permits solution of problem of considerably larger dimensionality.

The investigated approach can be extended to the solution of inverse problems with movable boundaries [5, 8] also in nonlinear formulation.

NOTATION

a	is the thermal diffusivity;
b	is the right end of the interval in x;
$f(\tau)$	is the given temperature dependence at the point $x = b$;
$J(q)$	is the functional to be minimized;
k	is the number of iteration;
$Q(\tau)$	is the heat flux at the point $x = b$;
$q(\tau)$	is the desired heat flux;
$T(x, \tau)$	is the temperature;
$T^*(\tau)$	is the computed temperature dependence at the point $x = b$;
x	is the coordinate;
Fo	is the Fourier number;
λ	is the thermal conductivity;
ξ	is the dimensionless coordinate;
τ	is the time;
τ_m	is the end of the investigated time interval;
$\psi(x, \tau)$	is the solution of the conjugate boundary value problem;
Δ	is the increment of the corresponding quantities.

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